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Magnetisation discontinuity of the two-dimensional Potts model

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Abstract. The q-state two-dimensional ferromagnetic Potts model has a first-order transition for $q>4$, its spontaneous magnetisation having a jump discontinuity. The magnitude of this discontinuity is calculated exactly for the square, triangular and honeycomb lattices: it depends only on q and is the same for all three lattices.

1. Introduction and summary

Kim (1981) has considered the jump discontinuity ΔM in the spontaneous magnetisation of the two-dimensional q-state Potts model. (A definition of ΔM is given in § 2 of this paper.) He evaluated the first six terms in a series expansion of ΔM in powers of q^{-1}, obtaining

$$\Delta M = 1 - q^{-1} - 3q^{-2} - 9q^{-3} - 27q^{-4} - 82q^{-5} - \dots \tag{1}$$

These terms are the same for the square, triangular and honeycomb lattices, so Kim conjectured that ΔM is the same for these three lattices. Here it will be shown that this 'universality' property is a consequence of the star-triangle relation and is true even for anisotropic models. It will also be shown that ΔM can be evaluated exactly by using corner transfer matrices. The result is

$$\Delta M = \prod_{n=1}^{\infty} [(1-x^{2n-1})/(1+x^{2n})] = \prod_{n=1}^{\infty} [(1-x^{n})/(1-x^{4n})], \tag{2}$$

where, for $q>4$, x is defined by

$$q = x + 2 + x^{-1}, \qquad 0<x<1. \tag{3}$$

It is straightforward to verify from (2) and (3) that Kim's large-q series (1) is correct. To investigate the behaviour near $q=4$, one can use the mathematical identity

$$\prod_{n=1}^{\infty} (1-\mathrm{e}^{-2\pi n z}) = z^{-1/2} \exp[\pi(z-z^{-1})/12] \prod_{n=1}^{\infty} (1-\mathrm{e}^{-2\pi n/z}), \tag{4}$$

which is true for all complex numbers z with positive real part. Setting

$$x = \mathrm{e}^{-2\theta}, \tag{5}$$

it follows that

$$\Delta M = 2\exp[-(4\theta^2+\pi^2)/16\theta]\prod_{n=1}^{\infty}[1-\exp(-2\pi^2 n/\theta)]/[1-\exp(-\pi^2 n/2\theta)]. \tag{6}$$

Thus ΔM decreases from 1 to 0 as q decreases from ∞ to 4, and near $q=4$,

$$\Delta M \sim 2\exp[-\pi^2/8(q-4)^{1/2}]. \tag{7}$$

This result agrees with the renormalisation group calculation of Cardy *et al* (1980, equation (3.21)).

2. Definitions

The zero-field ferromagnetic Potts model on an arbitrary lattice $\mathscr{L}$ has Hamiltonian

$$\mathscr{H} = -\sum_{\langle ij\rangle} J_{ij}\delta(\sigma_i,\sigma_j), \tag{8}$$

where for each site i there is a 'spin' σ_i, with values $1, 2, \ldots, q$. The summation is over all edges $\langle ij\rangle$ of $\mathscr{L}$, and each J_{ij} is a positive real number.

The partition function is

$$Z = \sum_{\sigma}\exp(-\mathscr{H}/kT), \tag{9}$$

where k is Boltzmann's constant, T is the temperature, and the summation is over all values of all the spins. Thus if $\mathscr{L}$ has N sites, there are q^N terms in this summation.

Consider some particular site and choose $i=0$ thereat. The probability that the spin σ_0 will have value 1 is

$$\langle\delta(\sigma_0,1)\rangle = Z^{-1}\sum_{\sigma}\delta(\sigma_0,1)\exp(-\mathscr{H}/kT). \tag{10}$$

In terms of this we can define a 'local magnetisation'

$$M = [q\langle\delta(\sigma_0,1)\rangle - 1]/(q-1). \tag{11}$$

The Hamiltonian (8) has the symmetry property that it is unchanged by incrementing every spin by one (modulo q). Provided the boundary conditions do not break this symmetry, it follows that all q states of any particular spin are equally likely, so that

$$\langle\delta(\sigma_0,1)\rangle = q^{-1}, \qquad M=0. \tag{12}$$

However, let us fix all boundary spins to be in state 1. Then M will be positive for any finite lattice. If we then take the thermodynamic limit of a large lattice, in such a way that site 0 lies deep inside the lattice, infinitely far from the boundaries, then we might expect that M would be insensitive to the boundary conditions, so that we would regain the symmetry property (12).

For sufficiently high temperatures, precisely this happens; but if T is less than some critical value T_c, then the effect of the boundary conditions persists even in this limit, and M is strictly positive. The symmetry is 'spontaneously broken'.

Further, if q is sufficiently large (greater than 4 in two dimensions), then the transition at $T=T_c$ is first order: M jumps discontinuously from zero to a strictly positive value, so that

$$M = 0 \text{ for } T > T_c, \qquad M > 0 \text{ for } T \leq T_c. \tag{13}$$

Writing M as a function $M(T)$ of T, it follows that at T_c it has a jump discontinuity

$$\Delta M = M(T_c) = \lim_{T \to T_c^-} M(T). \tag{14}$$

This is the ΔM discussed by Cardy *et al* (1980) and Kim (1981). For a translation-invariant lattice model, it is independent of the position of site 0, provided of course that the site is infinitely far from the boundaries.

3. Universality

Kim (1981) rightly conjectured that ΔM was the same for the square, triangular and honeycomb lattices. As will now be shown, this is a consequence of the star-triangle relation.

We can adapt the Ising model argument of Baxter and Enting (1978), hereinafter referred to as BE. We replace the Ising model therein by a Potts model, allowing all spins σ_i to take the values $1, \ldots, q$. We still start by considering an anisotropic honeycomb lattice model of $2N$ sites, but now the interaction energy of an edge (i, l) is $-kTL_r\delta(\sigma_i, \sigma_l)$. Here r takes the values $1, 2, 3$ according to the direction of the edge (i, l). If J_r is the value of the J_{il} in (8), then

$$L_r = J_r/kT, \qquad r = 1, 2, 3. \tag{15}$$

We can regard L_1, L_2, L_3 as dimensionless interaction coefficients.

Now we consider a star i, j, k, l, l being the centre site, as in figure 1 of BE. We attempt to equate its Boltzmann weight (summed over σ_l) to that of a triangle i, j, k with dimensionless interaction coefficients K_1, K_2, K_3. We are free to introduce also a normalisation factor R, so we obtain the equation

$$\sum_{\sigma_l} \exp[L_1\delta(\sigma_i, \sigma_l) + L_2\delta(\sigma_j, \sigma_l) + L_3\delta(\sigma_k, \sigma_l)]$$
$$= R \exp[K_1\delta(\sigma_j, \sigma_k) + K_2\delta(\sigma_k, \sigma_i) + K_3\delta(\sigma_i, \sigma_j)]. \tag{16}$$

We want this star-triangle relation to be true for all values of $\sigma_i, \sigma_j, \sigma_k$. By considering the cases when they are all equal, two are equal, or none are equal, we find that (16) is equivalent to the five equations

$$q - 1 + z_1'z_2'z_3' = Rz_1z_2z_3, \qquad q - 2 + z_1' + z_2'z_3' = Rz_1, \qquad q - 2 + z_2' + z_3'z_1' = Rz_2,$$
$$q - 2 + z_3' + z_1'z_2' = Rz_3, \qquad q - 3 + z_1' + z_2' + z_3' = R, \tag{17}$$

where

$$z_j = \exp(K_j), \qquad z_j' = \exp(L_j) \tag{18}$$

for $j = 1, 2, 3$. We shall use the variables

$$v_j = z_j - 1, \qquad v_j' = z_j' - 1. \tag{19}$$

Eliminating R, z_1, z_2, z_3 between the equations (17) leaves the relation

$$q^2 + q(v_1' + v_2' + v_3') = v_1'v_2'v_3'. \tag{20}$$

Thus we can only apply the star-triangle relation to the honeycomb lattice Potts model when the interaction coefficients satisfy (20). Kim and Joseph (1974) and Baxter *et al* (1978) have shown that this is precisely the condition for the ferromagnetic model to be critical, i.e. for the temperature T to have its critical value T_c. This is the case we consider in this paper in our study of $\Delta M = M(T_c)$. The free energy, internal energy and latent heat of the two-dimensional critical Potts model have already been calculated (Baxter 1973, Baxter *et al* 1978, Baxter 1982a, b).

The last relation in (17) corresponds to $\sigma_i, \sigma_j, \sigma_k$ all being different. For the Ising model case, when $q = 2$, this cannot occur, so this relation disappears. We then no longer have (20), so the star-triangle relation can be applied to the Ising model for *all* temperatures T. However, here we are interested in arbitrary values of q, so can only consider the critical case $T = T_c$.

Comparing the last equation in (17) with (20), it is readily seen that $R = q^{-1}v_1'v_2'v_3'$. Subtracting the last equation in (17) from each of the middle three, it follows that

$$v_jv_j' = q, \qquad j = 1, 2, 3. \tag{21}$$

Thus (20) implies that

$$v_1v_2v_3 + v_2v_3 + v_3v_1 + v_1v_2 = q, \tag{22}$$

which is a constraint on the interaction coefficients K_1, K_2, K_3. Indeed, if the star-triangle relation is applied to all the N down-pointing stars of the honeycomb lattice, the effect is to convert the model to a Potts model on the triangular lattice, with interaction coefficients K_1, K_2, K_3, and with an extra factor R^N multiplying the partition function. Thus we are restricted to considering only triangular lattice models that satisfy (22), and again this is the condition for criticality.

Alternatively, it is shown in BE how we can successively use the star-triangle relation to convert the honeycomb lattice model to one on a lattice with a large central square-lattice region, having horizontal and vertical edge interaction coefficients K_3 and L_3. These are not arbitrary, but satisfy (21), i.e. $v_3v_3' = q$, which is the criticality condition for the square lattice Potts model.

All these transformations leave unchanged the spin on site i in figures 1 and 2 of BE. This spin is deep inside the lattice and we can fix the top and bottom boundary spins to have value 1. (The transformations do modify the boundary conditions, but by using row-to-row transfer matrices we can argue that these modifications should not affect the present argument.) The factors R affect the partition function, but cancel out of the correlation (10). It follows that $\langle\delta(\sigma_i, 1)\rangle$ must have the same value for all three lattices. Thus

$$(\Delta M)_{L_1,L_2,L_3} = (\Delta M)_{K_1,K_2,K_3} = (\Delta M)_{K_3,L_3}, \tag{23}$$

where the three ΔM are evaluated for the honeycomb, triangular and square lattices, respectively.

We can regard q as given, L_2 and L_3 as independent variables, and L_1 as defined by (20). Then the honeycomb lattice ΔM is a function of L_2 and L_3. From (23), using the fact that K_3 is determined from L_3 by (21), it follows that ΔM is independent of L_2. By symmetry it must also be independent of L_3. Thus

$$\Delta M = \text{function only of } q, \tag{24}$$

being the same for the honeycomb, triangular and square lattice critical ferromagnetic Potts models.

3.1. *Further generalisations*

This result can be further generalised, e.g. to some special inhomogeneous lattice models, by using the concept of 'Z-invariance' (Baxter 1978), applying it to the Potts model rather than the eight-vertex model. We shall not need such generalisations here, but let me briefly remark that this can be done by allowing the spins σ_l to take the values $1, \ldots, q$ and replacing the definition of $\mathscr{H}$ in equation (2.1) of Baxter (1978) by

$$-\beta\mathscr{H} = \sum [K_{PAR}\delta(\sigma_l, \sigma_n) + K_{RAQ}\,\delta(\sigma_m, \sigma_p)]. \tag{25}$$

Then $\mathscr{H}$ is the Hamiltonian of two Potts models, one on the shaded faces of the straight-line lattice in figure 1 of Baxter (1978), the other on the unshaded faces. The relation (4.3) of that paper then factors into two identical star-triangle relations. Provided these are satisfied for all triplets of straight lines, one can repeat the argument of pp 324–6 therein and establish as an analogue of (5.1) that $\langle\delta(\sigma_l, 1)\rangle$ is a function only of q, for any face l deep within the straight-line lattice.

4. Dichromatic polynomial

Because of (24), we can without loss of generality now restrict attention to a square lattice Potts model with horizontal interaction coefficient K and vertical interaction coefficient L, satisfying the ferromagnetic criticality condition

$$(\mathrm{e}^K - 1)(\mathrm{e}^L - 1) = q. \tag{26}$$

Setting

$$v = \mathrm{e}^K - 1, \qquad w = \mathrm{e}^L - 1, \tag{27}$$

we can write the partition function (9) as

$$Z = \sum_\sigma \prod [1 + v\delta(\sigma_i, \sigma_j)] \prod [1 + w\delta(\sigma_i, \sigma_k)], \tag{28}$$

where the first product is over all horizontal edges (i, j), the second is over all vertical edges (i, k).

We can write Z as a dichromatic polynomial (Kasteleyn and Fortuin 1969, Baxter 1973, Baxter *et al* 1976). Expand the products in (28), and draw a line on the corresponding edge of the lattice if one takes the $v\delta(\sigma_i, \sigma_j)$ or $w\delta(\sigma_i, \sigma_k)$ term corresponding to that edge, no line if one takes the term unity. Then there is a one-to-one correspondence between all graphs G on the lattice, and terms in the expansion. For each term the σ-summations can be performed. Remembering that every boundary site j is fixed to have $\sigma_j = 1$, we find that

$$Z = \sum_G q^C v^l w^m, \tag{29}$$

where the summation is over all graphs G (i.e. ways of drawing lines on the edges of the lattice), l is the number of horizontal lines, m is the number of vertical lines, and C is the number of connected components (including isolated sites) not containing any boundary sites.

We can regard the boundary sites as connected, and can regard them, and all sites connected to them, as forming the 'boundary cluster'. Then C is the number of connected components of G, not including the boundary cluster.

Applying the same procedure to the numerator in (10), we have to distinguish between graphs in which site 0 is connected to the boundary, and those in which it is not. We find that

$$\langle \delta(\sigma_0, 1) \rangle = P + q^{-1}(1-P), \tag{30}$$

where

$$P = Z^{-1} \sum_G{}' q^C v^l w^m \tag{31}$$

and the prime means that the summation is restricted to graphs in which 0 is connected to the boundary.

Substituting (30) into (11), we obtain

$$M = P. \tag{32}$$

Thus the zero-field magnetisation of the Potts model is the same as the 'percolation' probability that site 0 belongs to the boundary cluster (Kelland 1976).

5. Six-vertex model

Having converted the Potts model to a dichromatic polynomial problem, we now transform the latter to a six-vertex model.

The procedure for Z is given by Baxter *et al* (1976). First turn the Potts model lattice $\mathscr{L}$ through 45° so that K is the interaction coefficient for SW–NE edges, L for SE–NW edges. Take this to be the lattice of broken lines and open circles in figure 1.

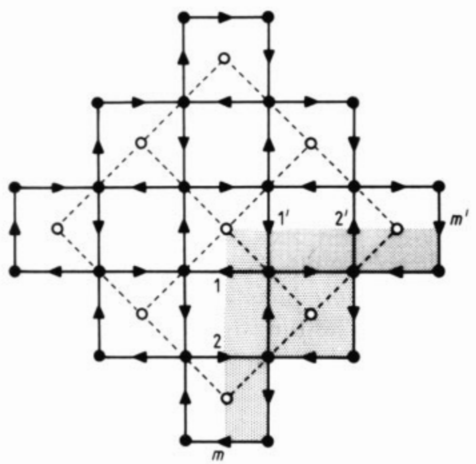

Figure 1. The Potts model lattice $\mathscr{L}$ (open circles and broken lines) and the corresponding six-vertex lattice $\mathscr{L}'$ (full circles and lines). The lower-right quadrant, corresponding to the $2^m \times 2^m$ corner transfer matrix A, is shown shaded. In this figure $m = 3$.

Remembering that the boundary sites are to be regarded as forming a cluster, but this cluster is not counted in the C of (29), we find that

$$Z = q^{(N-2)/2} Z_{6V}, \tag{33}$$

where N is the number of sites of $\mathscr{L}$, and Z_{6V} is the partition function of a six-vertex model on the lattice $\mathscr{L}'$ of full lines and circles.

As usual (Lieb 1967), this six-vertex model is obtained by placing arrows on the edges of the lattice so that at each site (or 'vertex') there are as many in-pointing arrows as out-pointing ones.

The sites of $\mathscr{L}'$ are either two-valent (i.e. have two neighbours), or are four-valent. The former occur on the outer perimeter of $\mathscr{L}'$: if an observer following the arrows turns to his left as he goes through such a site, then the site is given a Boltzmann weight $\exp(\theta/4)$; if he turns to the right, it is given a weight $\exp(-\theta/4)$. The parameter θ is defined by (3) and (5), so that

$$q^{1/2} = 2\cosh\theta, \qquad \theta > 0. \tag{34}$$

The four-valent sites of $\mathscr{L}'$ are of two types: those lying on SW–NE edges of $\mathscr{L}$, and those on SE–NW edges. Of these, the outermost ones lie on boundary edges of $\mathscr{L}$, while the rest lie on internal edges. At each site there are six possible configurations of arrows. Arranging them as in Lieb (1967) and Baxter (1973), we assign to them Boltzmann weights

$$\omega_1, \ldots, \omega_6 = a, a, b, b, as + bs^{-1}, as^{-1} + bs, \tag{35}$$

where

$$s = \exp(\theta/2) \tag{36}$$

and a, b have the values given in table 1. Then the six-vertex model partition function is

$$Z_{6V} = \sum \prod (\text{weights}), \tag{37}$$

where the sum is over all allowed ways of placing arrows on the edges of $\mathscr{L}'$, and for each such arrangement the product is of the Boltzmann weights of all the sites.

Table 1. Values of a and b in (35) for the four types of edges of the lattice $\mathscr{L}$, and for the corresponding sites of $\mathscr{L}'$.

		a	b
SW–NE	Internal	$q^{-1/2}v$	1
SW–NE	Boundary	$q^{-1/2}$	0
SE–NW	Internal	1	$q^{-1/2}w$
SE–NW	Boundary	0	$q^{-1/2}$

Kelland (1975) has shown that one can also express the percolation probability P in terms of the six-vertex model. Take 0 to be the centre site of the lattice $\mathscr{L}$ in figure 1. Label the horizontal edges of $\mathscr{L}'$ that are directly beneath 0 as $1, \ldots, m$, as in figure 1. Note that m must be odd. Let $\alpha_1, \ldots, \alpha_m$ be the 'arrow spins' on these edges: $\alpha_j = +1$ if the arrow on edge j points to the right, $\alpha_j = -1$ if the arrow points to the left. Write α for $\{\alpha_1, \ldots, \alpha_m\}$ and introduce the function

$$s(\alpha) = \exp[-(\theta + \tfrac{1}{2}\mathrm{i}\pi)(\alpha_1 + \alpha_2 + \ldots + \alpha_{m-1})] \tag{38}$$

and its expectation value

$$\langle s(\alpha)\rangle = Z_{6V}^{-1} \sum s(\alpha) \prod (\text{weights}). \tag{39}$$

As is shown in Baxter *et al* (1976), the dichromatic polynomial (29) can be written as a sum over polygon decompositions of $\mathscr{L}'$. Each connected component of G is surrounded by a polygon, and so is each circuit. In particular, since the boundary sites of $\mathscr{L}$ are connected, there is always a polygon on the perimeter of $\mathscr{L}'$. The six-vertex model arrows follow one another sequentially round each polygon: if the arrows go anticlockwise, then the polygon is given a weight $\exp(\theta)$; if clockwise, a weight $\exp(-\theta)$.

If the site 0 is connected to the boundary sites, then only the perimeter polygon can surround 0. Every other polygon must include an even number of the edges $1, \ldots, m-1$ of $\mathscr{L}'$, and for every polygon arrow covering there must be as many right-pointing arrows on these edges as there are left-pointing arrows. Thus $\alpha_1+\ldots+\alpha_{m-1}=0$ and $s(\alpha)=1$.

On the other hand, if site 0 is not connected to the boundary then there must be at least one polygon (in addition to the perimeter polygon) surrounding it. This will include an odd number of the edges $1, \ldots, m-1$. If the arrows go anticlockwise round this polygon, then there will be one more right-pointing arrow on the edges $1, \ldots, m-1$ than left-pointing ones. The corresponding contribution to the sum in (39) therefore contains not only the weight factor $\exp(\theta)$, but also a factor $\exp(-\theta-\frac{1}{2}\mathrm{i}\pi)$ coming from $s(\alpha)$. The combined weight is therefore $\exp(-\frac{1}{2}\mathrm{i}\pi)$. Conversely, if the arrows go clockwise round the polygon, we obtain a combined weight $\exp(\frac{1}{2}\mathrm{i}\pi)$. Summing these gives $2\cosh(\frac{1}{2}\mathrm{i}\pi)=0$.

It follows that the sum in (39) is the same as the primed sum in (31), and hence that

$$M=P=\langle s(\alpha)\rangle. \tag{40}$$

5.1. *Corner transfer matrices*

We can express $\langle s(\alpha)\rangle$ in terms of corner transfer matrices (Baxter 1980, 1981). In figure 1 is shown the ground-state energy configuration of arrows on $\mathscr{L}'$. This configuration is 'staggered', the arrows alternating in direction from edge to edge.

For each edge r of $\mathscr{L}'$, we can define a 'staggered arrow spin' μ_r such that $\mu_r=+1$ if the arrow on edge r has the same direction as in figure 1, while $\mu_r=-1$ if it has the opposite direction. Then for the horizontal edges labelled 1 to m in figure 1, we see that

$$\mu_r=(-1)^r\alpha_r, \qquad r=1, \ldots, m. \tag{41}$$

In figure 1 we have labelled as $1', 2', \ldots, m'$ the m vertical edges directly to the right of 0. Let $\mu_1', \ldots, \mu_m'$ be the corresponding staggered arrow spins. Write μ for $\{\mu_1, \ldots, \mu_m\}$, and μ' for $\{\mu_1', \ldots, \mu_m'\}$, and let $A_{\mu\mu'}$ be the partition function of the six-vertex model for the lower-right quadrant of $\mathscr{L}$, i.e.

$$A_{\mu\mu'}=\sum\prod(\text{weights}), \tag{42}$$

where the product is over all sites in the lower-right quadrant and the sum is over all allowed coverings of the adjacent edges, the arrows on the labelled edges in figure 1 being fixed to have spins $\mu_1, \ldots, \mu_m'$. If there is no allowed covering that is consistent with this choice of the labelled edge arrows, then $A_{\mu\mu'}=0$.

Similarly, let $\mu''(\mu''')$ be the staggered arrow spins on the edges directly above (to the left of) 0, and define $B_{\mu'\mu''}$ for the upper-right quadrant, $C_{\mu''\mu'''}$ for the upper-left, and $D_{\mu'''\mu}$ for the lower-left. Then by factoring the product in (37) into four terms, one for each quadrant, and summing over arrow coverings of edges within each quadrant, we obtain

$$Z_{6V} = \sum_{\mu} \sum_{\mu'} \sum_{\mu''} \sum_{\mu'''} A_{\mu\mu'} B_{\mu'\mu''} C_{\mu''\mu'''} D_{\mu'''\mu}. \tag{43}$$

Obviously, we can regard $A_{\mu\mu'}$ as the element (μ, μ') of a $2^m \times 2^m$ matrix A, and similarly for B, C, D. We can then write (43) as

$$Z_{6V} = \mathrm{Tr}\, ABCD. \tag{44}$$

Let S be the diagonal $2^m \times 2^m$ matrix whose entries are the $s(\alpha)$ given by (38) and (41), i.e. its entry in position (μ, μ) is

$$S_{\mu\mu} = \exp[(\theta + \tfrac{1}{2}\mathrm{i}\pi) g(\mu)], \tag{45}$$

where

$$g(\mu) = \mu_1 - \mu_2 + \mu_3 - \ldots - \mu_{m-1}. \tag{46}$$

Then (39) can be written in the simple form

$$\langle s(\alpha) \rangle = \mathrm{Tr}\, SABCD \,/\, \mathrm{Tr}\, ABCD. \tag{47}$$

The ice rule ensures that there should be as many left-pointing arrows on the edges $1, \ldots, m$ as there are down-pointing arrows on $1', \ldots, m'$. Since m and m' both lie on the perimeter polygon, $\mu_m = \mu'_m$. It follows that $g(\mu) = g(\mu')$, and hence that S commutes with A. Similarly, S commutes with B, C, D. We can readily establish the symmetry relations

$$A = C = A^{\mathrm{T}} = C^{\mathrm{T}}, \qquad B = D = B^{\mathrm{T}} = D^{\mathrm{T}}, \tag{48}$$

so from (40) and (47) it follows that

$$M = \mathrm{Tr}\, S(AB)^2 \,/\, \mathrm{Tr}\, (AB)^2. \tag{49}$$

6. Thermodynamic limit

The equivalences of §§ 4 and 5 are exact even for finite lattices, and do not depend on the criticality condition (26).

Let us now use (26), i.e. $vw = q$. From table 1, this implies that the ratio a/b is the same for all internal sites of $\mathscr{L}$, having the value

$$x = q^{-1/2} v = q^{1/2} w^{-1}. \tag{50}$$

Apart from trivial renormalisations of $\omega_1, \ldots, \omega_6$, it follows that these weights are the same for all internal sites of $\mathscr{L}$, so that the six-vertex model is homogeneous.

Further, there must be as many vertices of type 5 as there are of type 6, being sources and sinks, respectively, of vertical arrows. We can therefore multiply all weights ω_5, and divide all weights ω_6, by $[(x + s^2)/(xs^2 + 1)]^{1/2}$. This leaves (37) and (39) unchanged, but the weights of the internal sites are now given, not by (35), but by

$$\omega_1, \ldots, \omega_6 = a, a, b, b, c, c, \tag{51}$$

where

$$c = [a^2 + b^2 + 2ab \cosh \theta]^{1/2}. \tag{52}$$

We can now regard the corner transfer matrices A, B, C, D as defined by (42), using the weights given by (51), with $a/b = x$. Apart from boundary conditions, the model is now the homogeneous zero-field six-vertex model. This is a special case of the zero-field eight-vertex model, for which the diagonal forms of A, B, C, D have been calculated (Baxter 1980, hereinafter referred to as 8V) in the thermodynamic limit.

We can use these results to evaluate (49), but we have to translate from our 'electrical' terminology of arrows on edges to the 'magnetic' terminology of Ising spins on faces. Take the arrow ground state of figure 1 to correspond to the ferromagnetic ground state (all spins up) of 8V. Write $K_{8\mathrm{V}}, L_{8\mathrm{V}}, M_{8\mathrm{V}}$ for the K, L, M of 8V. We can relate these to our weights a, b, c by using the Ising spin formulation of the eight-vertex model (Kadanoff and Wegner (1971): since we are using the staggered arrow state of figure 1 as a reference, their a, b, c, d correspond to our $c, 0, a, b$). We find that

$$\begin{aligned} a:b:c:0 = {} & \exp(K_{8\mathrm{V}} - L_{8\mathrm{V}} - M_{8\mathrm{V}}) : \exp(-K_{8\mathrm{V}} + L_{8\mathrm{V}} - M_{8\mathrm{V}}) : \\ & \exp(K_{8\mathrm{V}} + L_{8\mathrm{V}} + M_{8\mathrm{V}}) : \exp(-K_{8\mathrm{V}} - L_{8\mathrm{V}} + M_{8\mathrm{V}}). \end{aligned} \tag{53}$$

Using equations (3.15) and (3.11) of 8V, it follows that the parameters q, λ, u therein are given by $q = 0$ and

$$a : b : c = \sinh(\lambda - u) : \sinh u : \sinh \lambda . \tag{54}$$

From (52), it follows that

$$\lambda = \theta. \tag{55}$$

The Ising spins $\sigma_1, \sigma_2, \ldots$ in (4.37) of 8V are related to our arrow spins $\mu_1, \mu_2, \ldots$ by

$$\mu_r = \sigma_r \sigma_{r+1}. \tag{56}$$

In the limit of a large lattice, i.e. when m is large, it is shown in 8V that the matrices A and B commute, so can be simultaneously diagonalised. From (4.30) and (4.37) of 8V (adapted to our boundary conditions), they then have diagonal entries

$$A_{\mu\mu} = \alpha \exp[\tfrac{1}{2} u h(\mu)], \qquad B_{\mu\mu} = \beta \exp[\tfrac{1}{2}(\theta - u) h(\mu)] \tag{57}$$

where α and β are scalar factors (independent of μ), and

$$h(\mu) = \mu_1\mu_2 + 2\mu_2\mu_3 + 3\mu_3\mu_4 + \ldots + (m-2)\mu_{m-2}\mu_{m-1} + (m-1)\mu_{m-1} - \mu_m. \tag{58}$$

(This result is exact in the sense that it correctly gives the r largest eigenvalues of $ABCD$, in the limit when we hold r fixed and let $m \to \infty$. For any r there is an integer p, independent of m, such that these eigenvalues are given by (57) and (58), with $\mu_p = \mu_{p+1} = \ldots = \mu_{m-1} = +1$. Our boundary conditions are such that μ_m plays a special, rather trivial, role, being the arrow spin of the perimeter polygon.)

The matrix S is unaffected by the similarity transformation that reduces A, B to the diagonal form (57). We can therefore use these diagonal forms, together with (45), in the expression (49) for the zero-field magnetisation M. Writing M as a function

$M(T)$ of T, and remembering that the results of this section apply only for $T=T_c$, it follows that

$$M(T_c)=\sum_{\mu}\exp[(\theta+\tfrac{1}{2}\mathrm{i}\pi)g(\mu)+\theta h(\mu)]\Big/\sum_{\mu}\exp[\theta h(\mu)], \tag{59}$$

both summations being over all values ($+1$ and -1) of $\mu_1,\ldots,\mu_m$. Note that α,β and u have all cancelled out of (59), so that $M(T_c)$ is a function only of θ, i.e. of q, in agreement with (24) and (14).

The denominator in (59) is readily calculated and is found to be

$$\exp[\tfrac{1}{2}(m^2-m+2)\theta](1+x)\prod_{j=1}^{m-1}(1+x^j), \tag{60}$$

where x is defined by (5). The numerator is calculated inductively in the Appendix: it is

$$\exp[\tfrac{1}{2}(m^2-m+2)\theta](1+x)\prod_{j=1}^{(m-1)/2}(1-x^{4j-2}). \tag{61}$$

Substituting these expressions into (59) and rearranging the products, it follows that

$$M(T_c)=\prod_{j=1}^{(m-1)/2}[(1-x^{2j-1})/(1+x^{2j})]. \tag{62}$$

Taking the limit $m\to\infty$, and using (14), we obtain the result given in equation (2).

Appendix

Performing the summation over μ_m, the numerator of (59) can be written as

$$2\cosh\theta\, R_{m-1}(+), \tag{A1}$$

where, for all positive integers n,

$$R_n(\mu_{n+1})=\sum\exp\left(\sum_{j=1}^{n}[(\theta+\tfrac{1}{2}\mathrm{i}\pi)(-1)^{j-1}\mu_j+j\theta\mu_j\mu_{j+1}]\right), \tag{A2}$$

the outer summation being over all values ($+1$ and -1) of $\mu_1,\ldots,\mu_n$.

One can immediately establish the recursion relation

$$R_n(\alpha)=\sum_{\beta}\exp[(\theta+\tfrac{1}{2}\mathrm{i}\pi)(-)^{n-1}\beta+n\theta\alpha\beta]R_{n-1}(\beta), \tag{A3a}$$

the summation being over $\beta=+1$ and -1. Together with the initial values

$$R_0(+)=R_0(-)=1, \tag{A3b}$$

this defines $R_n(+)$ and $R_n(-)$ for all non-negative integers n. By direct substitution, one can verify that the solution of (A3) is

$$R_{2p-1}(+)=\mathrm{i}x^{(2p-1)/2}F_p,\qquad R_{2p-1}(-)=0,\qquad R_{2p}(+)=F_p,\qquad R_{2p}(-)=x^{2p}F_p, \tag{A4}$$

where x is defined by (5) and

$$F_p=\exp[p(2p+1)\theta]\prod_{j=1}^{p}(1-x^{4j-2}). \tag{A5}$$

Remembering that m is odd, we obtain the result (61) for the expression (A1).

The arrow configuration shown in figure 1 is the ground-state configuration of the six-vertex model, provided $T < T_c$. For $T > T_c$, the ground-state configuration is that obtained from figure 1 by reversing all arrows.

At $T = T_c$, the infinite system, with weights given by (51), is unchanged by reversing all arrows. Thus both the above-mentioned configurations are contenders for the ground state. However, the boundary conditions are such as to favour the former configuration. For this reason, what we have calculated in (59) is the limit of $M(T)$ as T tends to T_c from below.

It is worth noting that we can also easily calculate the limit of $M(T)$ as T tends to T_c from above. To do this, we evaluate the eigenvalues of A and B for the case when the second arrow configuration is the ground state. The effect of this is simply to negate the spins $\mu_1, \ldots, \mu_{m-1}$ on the RHS of (58). (The perimeter spin μ_m plays a trivial role, merely contributing a factor $2\cosh\theta$ to both the numerator and denominator of (59): it is irrelevant whether or not we negate μ_m.) This leaves the denominator of (59) unchanged, but replaces $R_{m-1}(+)$ by $R_{m-1}(-)$ in the expression (A1) for the numerator. From (A4), this introduces an extra factor x^{m-1} into (A1), and hence into (62). Remembering that $|x| < 1$, and taking the limit $m \to \infty$, it follows that

$$\lim_{T \to T_c^+} M(T) = 0, \tag{A6}$$

in agreement with (13).

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